Sylvain Miossec
Yannick Aoustin

Institut de Recherche en Communications et
Cybernétique de Nantes, U.M.R. 6597,
1 rue de la Noë, BP 92101
44321 Nantes cedex 3, France
Sylvain.Miossec@irccyn.ec-nantes.fr
Yannick.Aoustin@irccyn.ec-nantes.fr

A Simplified Stability Study for a Biped Walk with Underactuated and Overactuated Phases

Abstract

This paper is devoted to a stability study of a walking gait for a biped. The walking gait is periodic and it is composed of a single-support phase, a passive impact, and a double-support phase. The reference trajectories are described as a function of the shin orientation versus the ground of the stance leg. We use the Poincaré map to study the stability of the walking gait of the biped. We only study the stability of dynamics not controlled during the single-support phase, i.e., the dynamics of the shin angle. We then suppose there is no perturbation in the tracking of the references of the other joint angles of the biped. The studied Poincaré map is then of dimension one. With a particular control law in double support, it is shown theoretically and in simulation that a perturbation error in the velocity of the shin angle can be eliminated in one step only. The zone of convergence in one step is determined. The condition of existence of a cyclic gait is given, and for a given cyclic gait, the stability condition is also given. It is shown that due to the given control law for the overactuated double-support phase, a cyclic motion is practically guaranteed to be stable. It should be noted it is possible for the biped to reach a periodic regime from a stopped position in one step.

KEY WORDS—walking biped, orbital stability, passive impact, dynamically stable gait, Poincaré return map, simulation results

1. Introduction

The stability of bipedal locomotion is currently an active research area. In the case of a biped with feet, some authors introduce criteria based on no knocking over around the extremities of the feet during the walking gait (see, for example, Goswami 1999; Vukobratovic et al. 1990). Generally, the sole of the support foot has its surface completely in contact with the ground (see, for example, Hirai et al. 1998; Löffler, Gienger, and Pfeiffer 2003). During the gait there is no underactuated phase for the biped.

To understand more deeply the effects of gravity on bipedal locomotion, some authors have studied planar bipeds without feet for a dynamically stable gait composed of single-support phases and passive impacts (see, for example, Grishin et al. 1994; Roussel, Canudas-de-Wit, and Goswami 1998; Aoustin and Formal'sky 2003; Chevallereau, Formal'sky, and Djoudi 2003; Chevallereau et al. 2003). The main difficulty is that in the single-support phase, the biped is underactuated. There are more degrees of freedom than actuators. Then, during this phase there is no equilibrium point, which is asymptotically stable. The actuated joints define a configuration for the biped, at each time. However, the biped's equilibrium is mainly connected to gravity force. To study the stability of such bipeds, it is necessary to introduce the concept of orbital asymptotical stability, usually studied with the Poincaré method. The Poincaré method consists of studying the stability of the Poincaré return map, the application, which gives the intersection of the orbit with a surface for a given previous intersection point. In Cheng and Lin (1996), a linearization of the Poincaré return map is analytically calculated. In Goswami, Espiau, and Kermane (1997), the linearization is numerically calculated. In Grizzle, Abba, and Plestan (2001), this concept of orbital stability and the Poincaré method was extended to systems with impulse effects, considering that the impact of the swing leg with the ground is usually impulsive. In Grizzle, Abba, and Plestan (2001), the dimension of the Poincaré return map is reduced to one. This fact greatly simplifies the study. Some other authors increased the convergence to the periodic orbit by modifying the step length or the inclination of the trunk from one step to the next step (see Aoustin and Formal'sky 2003). Other authors used a double-support phase, which also allows the stability to be increased (see Grishin et al. 1994; Zonfrilli, Oriolo, and Nardi 2002). In

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Grishin et al. (1994), a bipedal dynamic gait with a single support and a double support is designed for a five-link biped mechanism, developed in the Moscow Lomonosov State University. From their experiments, the authors show that the reference motion of the biped in single support can be described as a simple inverted pendulum, and they choose the final configuration of the double-support phase, to design a cyclic gait in spite of tracking errors. In Zonfrilli, Oriolo, and Nardi (2002), a walking gait with single-support and double-support phases is designed for a biped without feet. This reference gait is tested with the robot Sony AIBO. With this strategy, using a double-support phase for which the biped is overactuated, they are able to start from a stop phase.

A stability study is proposed in this paper for a walking biped with an underactuated single-support phase and a non-instantaneous double-support phase. The stability study is restricted to a one-dimensional space by using the Poincaré return map and supposing that the actuated joint reference trajectories are exactly followed. The main idea in the definition of the reference trajectory is that the actuated joint variables are polynomial functions in the absolute orientation angle of the biped. With an appropriate control law in double support, we show how this phase can improve the stability of the walk. For example, we determine the conditions of existence of a cyclic stable motion with double support. In the case of a cyclic stable motion, we determine a zone in the Poincaré return map, where the convergence from the current bipedal's state to the cyclic nominal gait is realized in one step. Also, we give a graphical representation that allows us to choose among an infinite number of possible motions, obtained for different dynamics of the absolute orientation angle of the biped.

The paper is organized as follows. The planar model and its dynamical model are presented in Section 2. In Section 3 we describe the reference motion and a reduced dynamic model associated with this reference motion. The control of the double-support phase is presented in Section 4. The stability study is proposed in Section 5. The simulation results are presented in Section 6. Finally, Section 7 contains our conclusion and perspectives.

2. Presentation of the Planar Biped and Its Dynamical Model

2.1. Biped Presentation

The object of our study is a five-link biped prototype, RABBIT, which is the result of a joint effort by several French research laboratories, grouped in a ROBEA project from CNRS (Chevallereau et al. 2003). RABBIT is currently developed in Grenoble. It is conceived to be the simplest mechanical structure that is still representative of human walking. RABBIT is guided around a central column by means of a radial bar. Its motion can then be considered restricted to the vertical sagittal plane. It has a trunk, two identical legs with knees, but no feet. There are four identical motors, which drive the haunches and the knees. The maximal value of their torque is 150 N m. Its control system comprises a computer, hardware servo-systems, and power amplifiers. The parameters of the four actuators with their gearbox reducers are specified in Table 1. The lengths, masses, location of the center of mass, and inertia moment of each link of RABBIT are collected in Table 2 with the help of Figure 1. Each actuated joint is equipped with one encoder to measure the joint position.

Figure 1 presents a diagram of the studied biped with some notations. We note that $\Gamma = [\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4]^{\mathrm{T}}$ is the torque vector, $\delta = [\delta_1, \delta_2, \delta_3, \delta_4]^{\mathrm{T}}$ are the actuated joint variables, $q = [\alpha, \delta^{\mathrm{T}}]^{\mathrm{T}}$ are the joint variables with the orientation of the biped in space, and $X = [q^{\mathrm{T}}, x_t, y_t]^{\mathrm{T}}$ are the configuration, orientation, and position vectors, where (x_t, y_t) is the position of the center of gravity of the trunk. Parameters s_i $(i = 1, 5)$ determine the distance between the centers of mass C_i $(i = 1, 5)$ of the shins and the knee joints for each leg. Parameters s_i $(i = 2, 3, 4)$ determine the distance between the centers of mass C_i $(i = 2, 3, 4)$ of the trunk, both thighs, and the hip joint. $R_i = [R_{ix}, R_{iy}]^{\mathrm{T}}$ $(i = 1, 2)$ are the ground reaction forces on feet 1 and 2, respectively.

2.2. Dynamic Model

The dynamic model is determined from the Lagrange equations. Its general form is

$$A(q)\ddot{X} + H(q, \dot{q}) = D_\Gamma \Gamma + D_1(q)R_1 + D_2(q)R_2. \qquad (1)$$

The inertia matrix $A(7 \times 7)$ of the biped is symmetric and positive definite. The centrifugal, Coriolis, and gravity effects are represented by the vector $H(7 \times 1)$. The torque vector Γ is taken into account by the fixed matrix $D_\Gamma(7 \times 4)$, consisting of zeros and units. Each matrix $D_j(7 \times 2)$ $(j = 1, 2)$ is a Jacobian matrix between foot velocity and allows us to take into account the ground reaction on each foot. If the point foot j is in the air, then $R_j = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$.

If the point foot j is in contact with the ground, the position variables X, the velocity variables $\dot{X}$, and the acceleration variables $\ddot{X}$ are constrained. In order to write these relations, we define the position, velocity, and acceleration of the point foot j in an absolute frame. The position of the point foot j is denoted $d_j(X)$. By differentiation of $d_j(X)$ we obtain the relation between the velocity $V_j = (V_{jx} \;\; V_{jy})^{\mathrm{T}}$ of the point foot j, and $\dot{X}$, given by

$$V_j = D_j(q)^{\mathrm{T}}\dot{X} \qquad (j = 1, 2). \qquad (2)$$

$D_j(7 \times 2)$ $(j = 1, 2)$ are the same matrices as in eq. (1).

Using another differentiation, we obtain the relation between the acceleration $\dot{V}_j = (\dot{V}_{jx} \;\; \dot{V}_{jy})^{\mathrm{T}}$ of the point foot j, and $\ddot{X}$, given by

$$\dot{V}_j = D_j(q)^{\mathrm{T}}\ddot{X} + H_{cj}(q, \dot{q}) \qquad (j = 1, 2). \qquad (3)$$

Table 1. Parameters of Actuators

Brushless Motor + Gearbox in Haunch and Knee	Mass (kg)	Gearbox Ratio	Rotor Inertia (kg m^2)
	2.82	50	3.32×10^{-4}

Table 2. Parameters of RABBIT

Physical Parameters of Each Body	Mass (kg)	Length (m)	Center of Mass Locations (m)	Moment of Inertia (kg m^2) Around the Center of Mass, C_i ($i = 1, \ldots, 5$)
Bodies 1 and 5: shin	3.32	0.40	$s_1 = s_5 = 0.127$	$I_1 = I_5 = 0.0484$
Body 3: trunk + actuators in each haunch	16.3	0.625	$s_3 = 0.13$	$I_3 = 1.9445$
Bodies 2 and 4: thigh + actuators in knee	4.1	0.40	$s_2 = s_4 = 0.163$	$I_2 = I_4 = 0.0693$

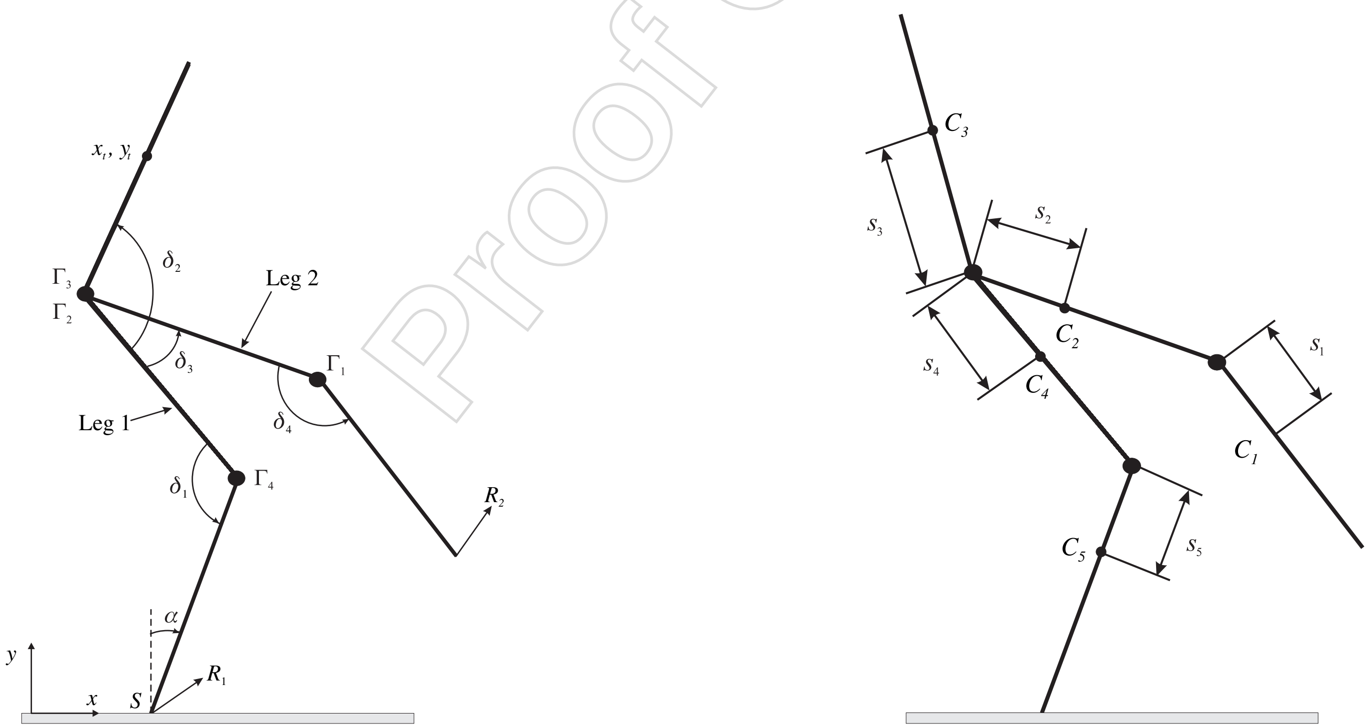

Fig. 1. Diagram of the five-link biped: generalized coordinates, torques, forces applied to the leg tips, center of mass locations.

$H_{cj}(2 \times 1)$ is equal to

$$\frac{\partial D_j^{\mathrm{T}} \dot{X}}{\partial q}\dot{q}.$$

Then the contact constraints for the point foot j with the ground are given by the three vector–matrix equations:

$$\begin{cases} d_j(X) = \text{constant} \\ V_j = D_j(q)^{\mathrm{T}}\dot{X} = 0 \qquad (j = 1, 2). \\ \dot{V}_j = D_j(q)^{\mathrm{T}}\ddot{X} + H_{cj}(q, \dot{q}) = 0 \end{cases} \tag{4}$$

These vector–matrix equations (4) mean that the position of the point foot j remains constant, and then that the velocity and acceleration of the point foot j are null.

3. Definition of a Reference Motion and Resulting Simplified Model

3.1. General Considerations

To design a walking gait for a biped with finite length for the feet, we have to take into account torque limitations in its ankles. Then, a biped without feet, such as RABBIT, can be viewed as a limit case when the length of the feet tends to zero. When studying the generation of walking motions for a biped without feet with a double-support phase, we face two problems: it is more difficult to obtain the double-support phase than to obtain a new single-support phase at the impact, and a walk with double support seems to consume more energy than a walk with instantaneous double support. However, the study of a walk with double support for a biped without feet is interesting for two reasons. First, the overactuated double-support phase improves the conditions of the orbital stability of the walk, as we see in the stability study, the object of this paper. This property arises, for example, as a quicker convergence, and even in one step, and as a larger attraction domain, and even a convergence from a stop position. Secondly, the study of the walk of the biped robot RABBIT is a first stage before studying the walk for a biped with feet, for which the double-support phase is obvious. Indeed, for an anthropomorphic walking gait, the double-support phase usually exists during a cyclic gait (see Devy et al. 2001). So the results obtained for a walk with a double-support phase for a simple biped without feet will be available for the case of a biped with feet.

We consider a walk with a single-support phase, an impact, and a double-support phase. For this biped, the single-support phase is underactuated (five degrees of freedom and four actuators). The strategy used is then to prescribe the reference trajectories of the four joint variables δ_j $(j = 1, ..., 4)$ as a function of the orientation variable of the shin, α. So, for a given angle α and assuming the perfect tracking of the four joint variables δ_j $(j = 1, ..., 4)$, the biped configuration is completely determined in the single-support phase. The behavior of the angular variable α results in the dynamic equations of the underactuated biped, and depends on the evolution of the four joint variables δ_j $(j = 1, ..., 4)$. In the double-support phase, the biped has three degrees of freedom and is overactuated. It is then possible to prescribe the reference trajectories of only three variables: we choose δ_j $(j = 1, 2)$ as a function of α, similarly to the single-support phase, and the angular variable α is defined as a polynomial as a function of time. With these three variables, the biped's motion is completely determined in double support. Note that in the following sections of the paper the subscripts SS and DS denote single support and double support, respectively.

3.2. Single-Support Phase

The reference trajectories of actuated joints, $\delta_{i,SS}$ $(i = 1, ..., 4)$ are defined as polynomial functions of fourth order in α:

$$\begin{array}{l} \delta_{i,SS}(\alpha) = a_{i0} + a_{i1}\alpha + a_{i2}\alpha^2 + a_{i3}\alpha^3 + a_{i4}\alpha^4 \\ \quad (i = 1, ..., 4) \end{array} . \tag{5}$$

We have chosen fourth-order polynomials for $\delta_{i,SS}(\alpha)$ because we wanted to prescribe initial and final positions and velocities and an intermediate configuration for the biped. Furthermore, if $\delta_{i,SS}(\alpha)$ $(i = 1, ..., 4)$ are exactly tracked, we are sure to reach the final configuration for the final value of α.

The temporal evolution of α results from the dynamics of the biped. Supposing $\delta_{i,SS}$ $(i = 1, ..., 4)$ are exactly tracking $\delta_{i,SS}(\alpha)$ $(i = 1, ..., 4)$, given by eq. (5) and applying the theorem of the total angular momentum in S (the contact point between the stance leg tip of the biped and the ground; see Figure 1), and writing the expression of angular momentum, we can define a simplified dynamic model on α (6), which then describes the biped's motion in the single-support phase:

$$\begin{cases} \dot{\sigma} = -Mg\,(x_G(\alpha) - x_S) \\ \dot{\alpha} = \frac{\sigma}{f(\alpha)} \end{cases} . \tag{6}$$

Here, M is the biped mass, g is the acceleration of gravity, and $x_G(\alpha)$ and $x_S(\alpha)$ are respectively the horizontal component of the positions of the biped's mass center and of the foot of the stance leg. σ is the angular momentum around S. The second equation of eq. (6) comes from the expression of the angular momentum given by

$$\sigma = \sum_{k=1}^{4} f_k(\delta_i)\dot{\delta}_k + f_5(\delta_i)\dot{\alpha}. \tag{7}$$

Then, by assuming the exact tracking of the reference trajectories, coefficients $f_k(\delta_i)$ $(k = 1, ..., 5)$ only depend on the biped parameters and α, since $\delta_i = \delta_i(\alpha)$ $(i = 1, ..., 4)$ only depend on α, and since $\dot{\delta}_i = \dot{\delta}_i(\alpha, \dot{\alpha}) = \partial\delta_i(\alpha)/\partial\alpha\ \dot{\alpha}$

($i = 1, \ldots, 4$) obtained by time differentiation of eq. (5) only depend on α and $\dot{\alpha}$. Finally, eq. (7) can be written $\sigma = f(\alpha)\dot{\alpha}$, as in eq. (6).

3.3. Simplified Model for the Passive Impact

The impact is considered rigid, passive, instantaneous, with impulsive ground reactions, and with a null restitution coefficient. At the impact instant, an inversion of leg role is performed. Impact equations can be obtained through integration of the matrix motion equation (1) between the infinitesimal time, just before and just after impact, considering here that both feet remain fixed on the ground after impact. The torques supplied by the actuators at the joints, Coriolis, and gravity forces have finite values, and thus they do not influence an impact (see, for example, Formal'sky 1982, 1997). Consequently, the impact equations can be written in the following matrix form:

$$A(q)\left(\dot{X}^{+} - \dot{X}^{-}\right) = D_1(q) I_{R_1} + D_2(q) I_{R_2}. \qquad (8)$$

The notation "+" ("−") means just after (before) impact. The term $I_{R_j}\,(I_{R_{jx}}, I_{R_{jy}})$ is the vector of the magnitudes of the impulsive reaction in leg j ($j = 1, 2$). From some manipulations detailed in Appendix A of this full impact model with relation (4) and supposing that reference trajectories are exactly tracked, we can obtain the following relation:

$$\dot{\alpha}^{+} = b\dot{\alpha}^{-} \qquad (9)$$

Term b depends on inertia parameters, on the biped configuration at impact, and on $\partial\delta_i(\alpha)/\partial\alpha$ ($i = 1, \ldots, 4$) just before impact.

3.4. Double-Support Phase

We define the reference trajectories of $\delta_{i,DS} = \delta_{i,DS}(\alpha)$ ($i = 1, 2$) as polynomial functions of α:

$$\delta_{i,DS}(\alpha) = a_{i0} + a_{i1}\alpha + a_{i2}\alpha^2 + a_{i3}\alpha^3 \quad \text{(i = 1,2)}. \qquad (10)$$

Taking into account that the biped is overactuated, α is defined as a polynomial of time:

$$\alpha_{DS}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3. \qquad (11)$$

Then, the motion of the biped is defined by $\delta_{i,DS}$ ($i = 1, 2$) and $\alpha_{DS}(t)$. A third-order polynomial is chosen for $\delta_{i,DS}(\alpha)$ ($i = 1, 2$) and $\alpha(t)$, to prescribe initial and final positions and velocities for the biped. Since the biped is overactuated, supposing $\delta_{i,DS}(\alpha)$ ($i = 1, 2$) and $\alpha(t)$ given by eqs. (10) and (11) are exactly tracked, all the dynamics are fixed in double support. That is, all the accelerations, the torques, and the reactions can be determined from the choice of the evolution of $\delta_{i,DS}(\alpha)$ ($i = 1, 2$) and $\alpha(t)$. The complete model (1) in double support will be used to perform those calculations.

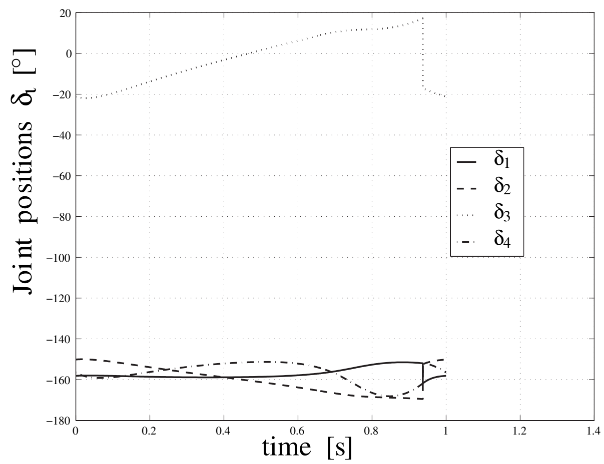

Fig. 2. Evolution of joint positions δ_1, δ_2, δ_3, and δ_4 as a function of time during one step, for the optimal motion considered in this paper. It is possible to observe the single support before the discontinuity due to exchange of legs, and the double support after the discontinuity.

3.5. Determining the Reference Motion

We have to find all the coefficients of the polynomial functions (5), (10), and (11). Taking into account that the motion is periodic and is continuous between each phase leads to relations between polynomial coefficients. It is then possible to reduce the number of parameters to 18 (see the principle of this reduction in Miossec and Aoustin 2002b). An optimization process is used to choose these 18 parameters so that an energy criterion is minimized and physical constraints are satisfied during the walk (such as no slipping, no take off of the feet, and limit torque). More details for the motion definition are given in Miossec and Aoustin (2002a) and Miossec (2004). In the following sections, we consider one set of polynomial coefficients obtained using this method. We represent in Figure 2 the evolution of the polynomials $\delta_{i,SS}(\alpha)$ ($i = 1, \ldots, 4$) and $\delta_{i,DS}(\alpha)$ ($i = 1, 2$) as $\delta_{i,DS}(\alpha)$ ($i = 3, 4$) obtained by the inverse geometric model, for the walking motion considered in this paper. We also represent in Figure 3 the evolution of torques needed to follow this motion, obtained by the inverse dynamic model. We can see that, for this cyclic motion obtained by optimization, the double support approximately represents 10% in time of a single support, for RABBIT.

4. Control of α_{DS} in Double Support

We only consider the control of α in double support, since we study only the dynamics of α. To only consider the dynamics of α, we consider that all actuated joints are exactly

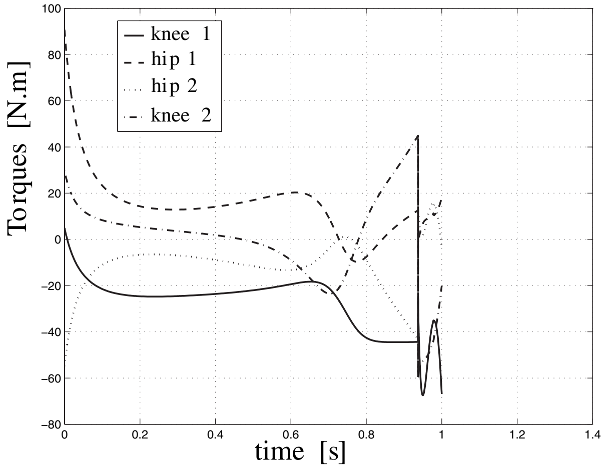

Fig. 3. Joint torques as a function of time for the optimal motion considered in this paper. Leg 1 is the support leg during single support and the front leg during double support.

tracking their reference motions.[1] The control law for α in double support that we use is a time optimal control. Indeed, it consists of applying the maximal acceleration $\ddot{\alpha}$ possible to converge to $\dot{\alpha}_c$ (the notation "c" designates the reference cyclic motion), and when $\dot{\alpha}_c$ is reached, to follow this cyclic reference motion. The expression of this control is

$$\ddot{\alpha} = \begin{cases} \ddot{\alpha}_{max}(\alpha, \dot{\alpha}) & if \quad \dot{\alpha}(\alpha) - \dot{\alpha}_c(\alpha) < 0 \\ \ddot{\alpha}_{min}(\alpha, \dot{\alpha}) & if \quad \dot{\alpha}(\alpha) - \dot{\alpha}_c(\alpha) > 0 \\ \ddot{\alpha}_c(\alpha, \dot{\alpha}) & if \quad \dot{\alpha}(\alpha) - \dot{\alpha}_c(\alpha) = 0. \end{cases} \tag{12}$$

Here, in eq. (12) $\ddot{\alpha}_{max}(\alpha, \dot{\alpha})$ and $\ddot{\alpha}_{min}(\alpha, \dot{\alpha})$ are, respectively, the maximal and minimal possible accelerations to satisfy the physical constraints (no slipping, no take off of the legs, torque limits). The same principle of control is used in Grishin et al. (1994) where a cyclic gait with a double-support phase and a single-support phase is designed for a biped with telescopic legs. In the double-support phase, the largest commanded acceleration possible is prescribed. They choose this maximal acceleration as a constant that allows the feet to remain on the ground.

Here we propose to determine these maximal and minimal accelerations at each instant using the dynamic model. We show the way to determine $\ddot{\alpha}_{max}(\alpha, \dot{\alpha})$ and $\ddot{\alpha}_{min}(\alpha, \dot{\alpha})$. The physical constraints considered are the following:

$$\begin{cases} R_{iy} \geq R_{iy,min} & (i = 1, 2) \\ -f_{max} R_{iy} \leq R_{ix} \leq f_{max} R_{iy} & (i = 1, 2) \\ -\Gamma_{max} \leq \Gamma_j \leq \Gamma_{max} & (j = 1, ..., 4). \end{cases} \tag{13}$$

$R_{iy,min} > 0$ is the minimal normal ground reaction. f_{max} is a maximal friction coefficient, which we choose for the ground. The inequality $f_{max} < f$ leads to the assumption to be inside the friction cone defined by f, the real limit friction coefficient. Our goal is first to write these constraints explicitly with respect to α, $\dot{\alpha}$, and $\ddot{\alpha}$, and then to extract the more restrictive constraints on $\ddot{\alpha}$.

Combining eqs. (4) and (10) into eq. (1) we obtain seven relations between α, $\dot{\alpha}$, $\ddot{\alpha}$, R_{ix}, R_{iy}, and Γ_j:

$$A_\alpha(\alpha)\ddot{\alpha} + H_\alpha(\alpha, \dot{\alpha}) = D_\Gamma \Gamma + D_1(\alpha) R_1 + D_2(\alpha) R_2. \tag{14}$$

The calculation of matrices A_α and H_α are detailed in Appendix B. We want to determine the eight unknown R_{ix}, R_{iy}, Γ_j as functions of α, $\dot{\alpha}$ and $\ddot{\alpha}$, whereas there are only seven equations in eq. (14). The biped is overactuated in double support. There are three degrees of freedom and there are four torques. Then, there are an infinity of solutions that we parametrize by R_{2x} (see Miossec and Aoustin 2002a or Miossec 2004 for a justification of this choice). So we obtain the following equations:

$$\left[R_{1x} R_{1y} R_{2y} \Gamma^{\mathrm{T}}\right]^{\mathrm{T}} = B(\alpha)\ddot{\alpha} + C(\alpha, \dot{\alpha}) + D(\alpha) R_{2x}. \tag{15}$$

The expressions of $B(\alpha)\ddot{\alpha}$, $C(\alpha, \dot{\alpha})$, and $D(\alpha)$ are presented Appendix C. With the seven equations (15) we can rewrite the 14 constraint inequations (13) as

$$E(\alpha)\ddot{\alpha} + F(\alpha, \dot{\alpha}) + G(\alpha) R_{2x} + P \leq 0. \tag{16}$$

The calculation of $E(\alpha)$, $F(\alpha, \dot{\alpha})$, $G(\alpha)$, and P is presented Appendix D.

$\ddot{\alpha}_{min}(\alpha, \dot{\alpha})$ and $\ddot{\alpha}_{max}(\alpha, \dot{\alpha})$ are then obtained by solving the two following simplex problems (see Dantzig 1963 for information about this type of problem):

$$\begin{cases} \ddot{\alpha}_{min}(\alpha, \dot{\alpha}) = \min\limits_{\ddot{\alpha}, R_{2x}} \ddot{\alpha} \\ E(\alpha)\ddot{\alpha} + F(\alpha, \dot{\alpha}) + G(\alpha) R_{2x} + P \leq 0 \\ \ddot{\alpha}_{max}(\alpha, \dot{\alpha}) = \max\limits_{\ddot{\alpha}, R_{2x}} \ddot{\alpha} \\ E(\alpha)\ddot{\alpha} + F(\alpha, \dot{\alpha}) + G(\alpha) R_{2x} + P \leq 0. \end{cases} \tag{17}$$

We now present a representative result of the control in the phase plane $(\alpha, \dot{\alpha})$ (see Figure 4; all angles of figures are in radian). For a given initial velocity, less in the module than

1. The reader can refer to Grizzle, Abba, and Plestan (2001), where the authors address the problem of a control of the actuated joints that can justify the restriction of the stability study to the dynamics which are not controlled in single support.

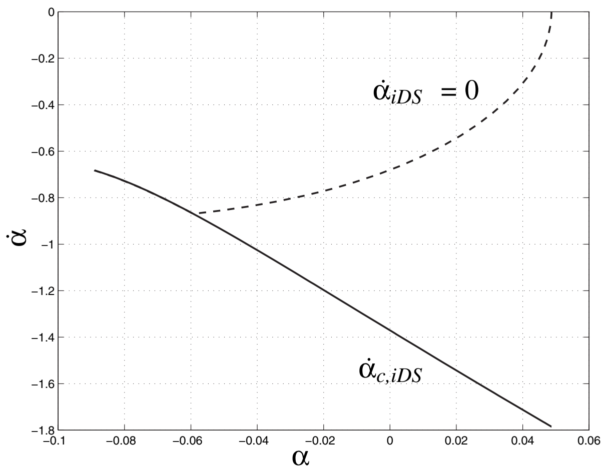

Fig. 4. Representation in the double-support phase of evolution of α for cyclic motion and for a converging motion starting from null velocity.

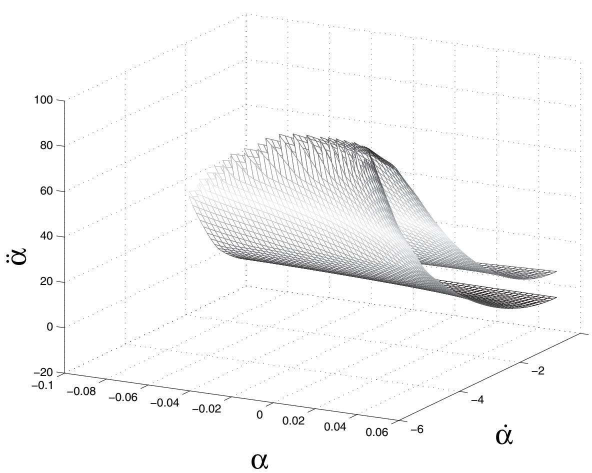

Fig. 5. Representation of $\ddot{\alpha}_{min}$ and $\ddot{\alpha}_{max}$ as functions of α and $\dot{\alpha}$.

the initial cyclic one, the minimal acceleration is applied and we can see in Figure 4 that the $(\alpha, \dot{\alpha})$ motion converges to the cyclic one. Then, after the intersection with the cyclic motion, it exactly follows the cyclic motion. We then have a null error at the end of the double-support phase.

Figure 5 presents the constraints $\ddot{\alpha}_{min}$ and $\ddot{\alpha}_{max}$ with respect to the two parameters α and $\dot{\alpha}$. These constraints $\ddot{\alpha}_{min}$ and $\ddot{\alpha}_{max}$ only depend on α and $\dot{\alpha}$. Figure 6 shows $\ddot{\alpha}_c$ for the cyclic motion and the corresponding evolution of $\ddot{\alpha}_{max}(\alpha, \dot{\alpha})$ and $\ddot{\alpha}_{min}(\alpha, \dot{\alpha})$. We must bear in mind that the real constraints $\ddot{\alpha}_{min}$

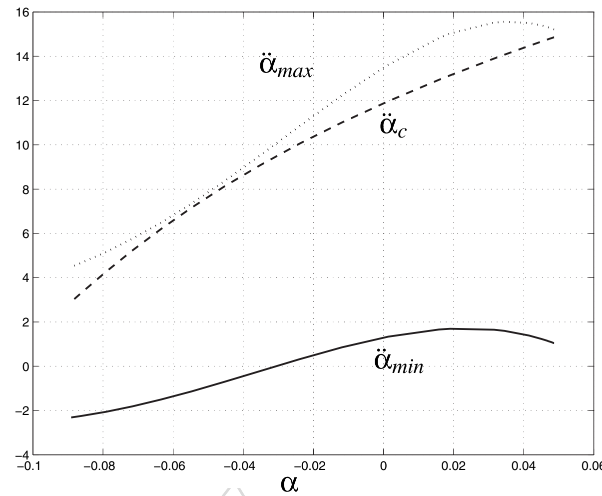

Fig. 6. Evolution of acceleration for the cyclic motion $\ddot{\alpha}_c$ with minimal and maximal accelerations possible $\ddot{\alpha}_{min}$ and $\ddot{\alpha}_{max}$ with respect to α.

and $\ddot{\alpha}_{max}$ depend on two parameters, and so in Figure 6 these constraints will change with the change of motion followed in double support.

5. Stability Study of the Reference Motion

We only study the stability of the dynamics of α. We suppose that the actuated joint trajectories are exactly followed (this assumption is a good approximation if we consider a sufficiently efficient control, without too many important perturbations). The choice of studying the stability of dynamics of α is due to the fact that it is the less stable dynamics, since it is the only dynamics not controlled during the single-support phase. The dynamics of α consist in eqs. (6) in single support, in discontinuity given by eq. (9) at impact and by α dynamics in double support depending on its control only, presented in Section 4. For this stability study, we use the Poincaré map.

We first present the Poincaré stability study in Section 5.1. Then in Section 5.2, for a desired final velocity $\dot{\alpha}_{fDS}$ we define a zone of the phase plan of double support that is a zone from which it is possible to converge to this desired final velocity. We also show that the control presented in Section 4 is a control that allows the convergence to a desired final velocity $\dot{\alpha}_{fDS}$ from the corresponding zone defined. In Section 5.3 we give some necessary and sufficient conditions for the existence of a cyclic motion depending on the dynamics of α, and we also give a sufficient condition of stability of such a cyclic motion with the control law presented in Section 4. When such a cyclic stable motion exists, we also show that the zone of finite time convergence in double support for the cyclic final velocity $\dot{\alpha}_{c,fDS}$ of a reference motion defines a one-step convergence zone. Finally, in Section 5.4 we present a graphical representation that allows us to see all the possible cyclic stable motions depending on the dynamics of α, and the corresponding domains of attraction.

All the results presented in this section are obtained under the following hypotheses.

(H1) $\ddot{\alpha}_{min}$ and $\ddot{\alpha}_{max}$ are always defined. This hypothesis means that it is always possible to satisfy constraints (13). In reality, to satisfy (H1) we consider a subspace where (H1) is satisfied. We can see in Figure 5 that the restriction applies for high values of $\dot{\alpha}$. Furthermore, we see in Section 5.4 that the more restrictive constraints are not those of double support but those of single support.

(H2) We consider the cases $\dot{\alpha} \leq 0$. The consequences of (H2) will be that α is always decreasing as a function of time, that the final velocity of double support $\dot{\alpha}_{fDS}$ is less than the initial velocity of double support $\dot{\alpha}_{iDS}$. Moreover, a motion of α in the phase plan representation will move from right to left.

(H3) $\ddot{\alpha}_{min} < 0$ when $\dot{\alpha} = 0$.

5.1. Presentation of Poincaré Stability Study

The Poincaré map consists of representing the evolution of the state of a cyclic motion at a characteristic instant of a system from one period to the following. For the walk considered here, with the previous assumptions, the Poincaré map is a function from only a one-dimensional space to a one-dimensional space (see also Aoustin and Formal'sky 2003; Grizzle, Abba, and Plestan 2001; Chevallereau, Formal'sky, and Djoudi 2003). In this paper we represent the Poincaré map of $\dot{\alpha}$ and we observe it at the beginning of the double-support or single-support phases. In the Poincaré map, a cyclic motion is represented by an invariant point, and this cyclic motion is stable if the slope at this point is between -1 and 1. In the following, we present the control of α in the double-support phase so that convergence to the cyclic motion is as fast as possible. Then we determine the double-support zone of convergence to a given final velocity of the double-support phase. Then, we deduce the conditions in α of the existence of a stable cyclic reference motion for a given final velocity of the double-support phase. Then, we present a method to see if a cyclic stable motion exists, and how to choose the more stable one, between different possible values of the final velocity of the double-support phase. Finally, we present the Poincaré map of $\dot{\alpha}_{iDS}$ and $\dot{\alpha}_{iSS}$ for the motion obtained by the optimization process presented in Section 3.5.

5.2. Determination of the Finite Time Convergence Zone in the Double-Support Phase

In this section we define the zone of convergence in one step in double support. We give a first definition:

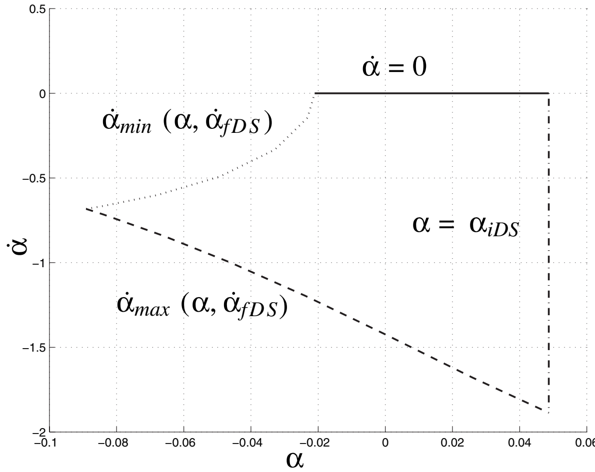

Fig. 7. Representation of the surface S_d in the phase plan.

DEFINITION 1. For a given velocity at the end of double support $\dot{\alpha}_{fDS}$, we consider the two functions $\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$ and $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$ defined as follows:

$$\begin{cases} \dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS}) = \displaystyle\int_{\alpha}^{\alpha_{fDS}} \frac{\ddot{\alpha}_{min}(\alpha, \dot{\alpha})}{\dot{\alpha}} ds + \dot{\alpha}_{fDS} \\ \dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS}) = \displaystyle\int_{\alpha}^{\alpha_{fDS}} \frac{\ddot{\alpha}_{max}(\alpha, \dot{\alpha})}{\dot{\alpha}} ds + \dot{\alpha}_{fDS} \end{cases} \tag{18}$$

$\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$ and $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$ are the evolutions of $\dot{\alpha}$ that give the final point $(\alpha_{fDS}, \dot{\alpha}_{fDS})$ when applying the minimal and maximal possible accelerations, respectively. These functions are obtained by reverse integration as a function of α. These functions are the solutions of the differential equations

$$\begin{aligned} \ddot{\alpha} &= \ddot{\alpha}_{min}(\alpha, \dot{\alpha}) \\ \ddot{\alpha} &= \ddot{\alpha}_{max}(\alpha, \dot{\alpha}). \end{aligned} \tag{19}$$

Now we can define a surface in the phase plane $(\alpha, \dot{\alpha})$ by closing the contour composed of $\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$ and $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$.

DEFINITION 2. For a given velocity at the end of double support $\dot{\alpha}_{fDS}$ we consider the surface S_d delimited by $\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$, $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$, $\dot{\alpha} = 0$ and $\alpha = \alpha_{iDS}$.

This surface is represented in Figure 7. It effectively constitutes a closed surface since $\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$ obviously intersects $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$ at the point $(\alpha_{fDS}, \dot{\alpha}_{fDS})$ and $\dot{\alpha} = 0$ also obviously intersects $\alpha = \alpha_{iDS}$ at the point $(\alpha_{iDS}, 0)$. Furthermore, we consider (H2); therefore, both $\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$ and $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$ will fatally intersect $\dot{\alpha} = 0$ or $\alpha = \alpha_{iDS}$.

The following proposition proves that whatever a point inside the surface S_d, it is possible to find a motion satisfying the constraints that allows us to converge to the point $(\alpha_{fds}, \dot{\alpha}_{fds})$.

PROPOSITION 1. $\forall(\alpha, \dot{\alpha}) \in S_d$, $\exists$ a motion $\alpha_{DS}(t)$ that goes from $(\alpha, \dot{\alpha})$ to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$ while still verifying constraints (13). Furthermore, $\forall(\alpha, \dot{\alpha}) \in \{(\alpha, \dot{\alpha})$, such that $(\alpha, \dot{\alpha}) \notin S_d$ *and* $\dot{\alpha} < 0$ and $\alpha < \alpha_{iDS}\}$, $\nexists$ $\alpha_{DS}(t)$ that allows us to go from $(\alpha, \dot{\alpha})$ to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$.

Proof. The demonstration is based on the construction of a motion that will converge to $(\alpha_{fds}, \dot{\alpha}_{fds})$. We design this motion as applying the maximal and minimal accelerations successively. We first apply $\ddot{\alpha}_{min}$. The motion is solution of the first differential equation of eq. (19). The function $\ddot{\alpha}_{min}$ satisfies the Lipschitz condition;[2] therefore, the two solutions are unique, and if they have different initial conditions, they will never meet. So the current solution will never intersect $\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS})$ or $\dot{\alpha} = 0$ since we supposed (H3), or $\alpha = \alpha_{iDS}$ since we supposed (H2). It will then intersect $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$. Finally, by applying $\ddot{\alpha}_{max}$ the motion will converge to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$. Of course, this motion is generally not the unique one that allows convergence for a given starting point inside S_d to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$. Usually there will be an infinity of such motions. If we now consider an initial point $(\alpha, \dot{\alpha}) \in \{(\alpha, \dot{\alpha})$, such that $(\alpha, \dot{\alpha}) \notin S_d$ and $\dot{\alpha} < 0$ and $\alpha < \alpha_{iDS}\}$ with the fact that two solutions of the same differential equation (19) with different initial conditions will never intersect, if $(\alpha, \dot{\alpha})$ is under S_d (above S_d) the solution by applying the maximal (minimal) acceleration possible will never reach $(\alpha_{fDS}, \dot{\alpha}_{fDS})$, otherwise it would mean that they have been an intersection with $\dot{\alpha}_{max}(\alpha, \dot{\alpha}_{fDS})$ $(\dot{\alpha}_{min}(\alpha, \dot{\alpha}_{fDS}))$, which is impossible.

The following proposition means that the convergence to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$ can be obtained with control (12) and with an appropriate reference motion.

PROPOSITION 2. If a reference motion $\alpha_c(t)$ such that when $\alpha_c = \alpha_{iDS}$, $\dot{\alpha}_c = \dot{\alpha}_{c,iDS}$ and when $\alpha_c = \alpha_{fDS}$, $\dot{\alpha}_c = \dot{\alpha}_{c,fDS}$ satisfy constraints (13) $\forall\alpha \in [\alpha_{fDS}, \alpha_{iDS}]$, then $\forall\alpha \in S_d$ the control law (12) allows convergence to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$.

Proof. First, such a $\alpha_c(t)$ motion is inside S_d since if it was not the case, it would not satisfy the constraints to arrive in $(\alpha_{fDS}, \dot{\alpha}_{fDS})$. Then by the same way as in the proof of Proposition 1, it can be shown that $\forall(\alpha, \dot{\alpha}) \in S_d$, control (12) will intersect $\alpha_c(t)$ and then converge to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$.

There exists an infinity of such reference cyclic motions $\alpha_c(t)$ satisfying the constraints and limit conditions $(\alpha_{iDS}, \dot{\alpha}_{iDS})$ and $(\alpha_{fDS}, \dot{\alpha}_{fDS})$. Also, the choice of this reference motion does not have any influence on the convergence to $(\alpha_{fDS}, \dot{\alpha}_{fDS})$ with this control.

5.3. Condition of Existence and Stability of a Cyclic Stable Walk

We have characterized the double-support phase in the previous section. In order to characterize the stability of the complete walk, we first characterize the single-support phase. Chevallereau, Formal'sky, and Djoudi (2003) obtained interesting results of stability for a gait composed of a single-support phase and an impact only. An analytic relation is written between the initial velocity of the biped and its velocity during the step. Then, from this result and taking into account the impact we can deduce the existence of an analytic relation between the velocity at the beginning of the double-support phase $\dot{\alpha}_{iDS}$ and the velocity at the end of the previous double-support phase $\dot{\alpha}_{fDS}$:

$$\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS}). \tag{20}$$

The calculation of function $a()$ is detailed in Appendix E.

Chevallereau, Formal'sky, and Djoudi (2003) also provide a condition of the existence of a periodic stable gait, and give the bounds of the convergence domain in single support. In particular, they found that the initial velocity of the single-support phase must be more than a minimum in order to avoid the biped falling back. They also provide a way to determine the maximal velocity allowed at the beginning of single support in order to satisfy ground constraints. We use both of these results in order to determine the domain of attraction of single support. We denote $\dot{\alpha}_{fDS,min}$ this lower bound and $\dot{\alpha}_{fDS,max}$ this upper bound.

From the previous results of both phases it is possible to obtain the following proposition.

PROPOSITION 3. For a given $\dot{\alpha}_{c,fDS} \in [\dot{\alpha}_{fDS,min}, \dot{\alpha}_{fDS,max}]$, a cyclic motion exists if and only if $\dot{\alpha}_{c,iDS} \in [\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}]$ where $\dot{\alpha}_{c,iDS} = a(\dot{\alpha}_{c,fDS})$ and $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{c,fDS}) = \dot{\alpha}_{min}(\alpha_{iDS}, \dot{\alpha}_{c,fDS})$ and $\dot{\alpha}_{iDS,max}(\dot{\alpha}_{c,fDS}) = \dot{\alpha}_{max}(\alpha_{iDS}, \dot{\alpha}_{c,fDS})$.

This proposition means that cyclic motions exist for $\dot{\alpha}_{c,fDS}$ if $\dot{\alpha}_{c,iDS}$ obtained from $\dot{\alpha}_{c,fDS}$, by the single support, impact and change of legs is inside the zone of double support that can converge to $(\alpha_{fDS}, \dot{\alpha}_{c,fDS})$.

Proof. For this proof we study the Poincaré map of $\dot{\alpha}_{fDS}$, i.e., we study the evolution of $\dot{\alpha}_{fDS}(n+1)$ with respect to $\dot{\alpha}_{fDS}(n)$, where n indicates the nth step. Let us consider $\dot{\alpha}_{c,iDS}(n+1) = a(\dot{\alpha}_{c,fDS}(n))$. First, to show the necessary and sufficient condition of existence of a cyclic motion, we consider the case when $\dot{\alpha}_{c,iDS}(n+1) \in]-\infty, \dot{\alpha}_{iDS,min}[\cup]\dot{\alpha}_{iDS,max}, 0[$. In this case, from Proposition 1 we know that we will not converge to $(\alpha_{fDS}, \dot{\alpha}_{c,fDS})$. The condition $\dot{\alpha}_{c,iDS}(n+1) \in [\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}]$ is then a necessary condition. In the case where $\dot{\alpha}_{c,iDS}(n+1) \in [\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}]$, from Proposition 1, a motion exists (for example, the cyclic motion used in the proof

2. Let us consider the domain defined by all the possible $(\ddot{\alpha}, R_{2x})$ that satisfy constraints (13). $\ddot{\alpha}_{min}$ will be the minimum value of $\ddot{\alpha}$ that belongs to this domain. The boundaries of this domain are linear functions of $\ddot{\alpha}$ and R_{2x}, whose coefficients are continuous in function of α and $\dot{\alpha}$, see eq. (13). Therefore, this domain is convex and continuous as a function of α and $\dot{\alpha}$. Then $\ddot{\alpha}_{min}$ is continuous in α and $\dot{\alpha}$ and satisfies the Lipschitz condition. The continuity of $\ddot{\alpha}_{min}$ and also $\ddot{\alpha}_{max}$ can be observed in Figure 5.

of Proposition 1) which allows us to reach $(\alpha_{fDS}, \dot{\alpha}_{fDS})$, and then to obtain a cyclic motion. The condition $\dot{\alpha}_{c,iDS}(n+1) \in [\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}]$ is then also a sufficient condition.

Furthermore, when a cyclic motion exists for a velocity $\dot{\alpha}_{fDS}$, there is an infinity of motions during the double support, which allow us to obtain this final velocity of double support $\dot{\alpha}_{fDS}$ from the initial velocity in double support $\dot{\alpha}_{iDS}$. We consider in the following only one cyclic motion in double support $\alpha_{cDS}(t)$ from the infinity of cyclic motions, defined by $\dot{\alpha}_{c,fDS}$ and $\dot{\alpha}_{c,iDS}$. This cyclic motion satisfies constraints (13). Such a cyclic motion is obtained by the optimization algorithm presented in Miossec and Aoustin (2002b), but other methods could be imagined to obtain a cyclic motion for α.

Proposition 4 gives a sufficient condition of stability of a cyclic motion.

PROPOSITION 4. Let $\alpha_{c,DS}(t)$ be a cyclic motion which satisfies eq. (13), if $\dot{\alpha}_{c,iDS} \in]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$, then the cyclic motion is stable with the control law (12).

Proof. To prove this proposition, we study the Poincaré map of $\dot{\alpha}_{fDS}$. Let us consider the case where $\dot{\alpha}_{c,iDS} \in]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$. Then it is always possible to find a closed interval for which there will be convergence to $\dot{\alpha}_{c,fDS}$ by the same control as in the proof of Proposition 1. For example, we can consider the interval $[(\dot{\alpha}_{iDS,min} + \dot{\alpha}_{c,iDS})/2, (\dot{\alpha}_{iDS,max} + \dot{\alpha}_{c,iDS})/2]$. For sufficiently small errors in the neighborhood of $\dot{\alpha}_{c,fDS}$ we have $\dot{\alpha}_{iDS}(n+1) \in [(\dot{\alpha}_{iDS,min} + \dot{\alpha}_{c,iDS})/2, (\dot{\alpha}_{iDS,max} + \dot{\alpha}_{c,iDS})/2]$ and then from Proposition 2 we have the convergence to $\dot{\alpha}_{c,fDS}$. We have then a local stability around the cyclic motion.

When $\dot{\alpha}_{c,iDS} = \dot{\alpha}_{iDS,min}$ or $\dot{\alpha}_{c,iDS} = \dot{\alpha}_{iDS,max}$ the cyclic motion is at the limit of the stable cyclic motions. In this case it is necessary to look at the slope of the Poincaré map at the cyclic point, to determine if there is stability. It is for this reason that we have to consider the open interval $]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$ for the stability condition. Furthermore, in all the cases where $\dot{\alpha}_{iDS} \in]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$ we obtain $\dot{\alpha}_{c,fDS}$ at the end of the double support, which corresponds to a case of convergence in one step. The sufficient condition of stability $\dot{\alpha}_{c,iDS} \in]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$ is not valid for all the control laws. For example, for a proportional derivative control law in double support, for an error at the beginning of double support, the final error at the end of double support will not be null. We do not obtain the convergence in one step, and we have to determine the slope of the Poincaré map to know if the cyclic motion is stable. However, generally, the slope of the Poincaré map will be close to a null slope. In the case of a finite time control and a cyclic motion, which satisfies the sufficient condition of stability $\dot{\alpha}_{iDS} \in]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$, we always have a cyclic stable motion and a zone of convergence in one step around this cyclic motion.

5.4. A Method to Visualize All the Cyclic Stable Walks

Generally, there will be an infinity of values of $\dot{\alpha}_{c,fds}$ for which there exists a cyclic stable motion. In this section we give a representation that will allow us to visualize all the possible cyclic stable motions, and will give a few stability criteria to choose among all these possibilities. These criteria will be in competition with the energy consumption criterion of the optimization process presented in Section 3.5.

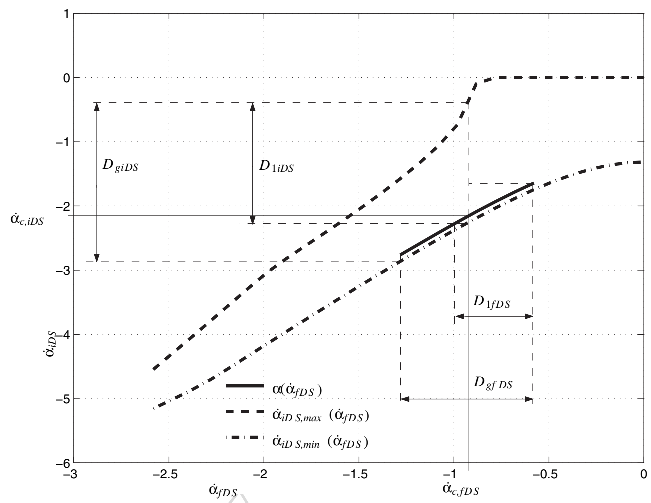

Fig. 8. Representation of $\dot{\alpha}_{iDS,max}(\dot{\alpha}_{fDS})$, $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{fDS})$ and $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$ with global attraction domain in $\dot{\alpha}_{fDS}$ D_{gfDS}, global attraction domain in $\dot{\alpha}_{iDS}$ D_{giDS}, one-step convergence domain in $\dot{\alpha}_{fDS}$ D_{1fDS}, one-step convergence domain in $\dot{\alpha}_{iDS}$ D_{1iDS}. The cyclic motion represented by point $(\dot{\alpha}_{c,fDS}, \dot{\alpha}_{c,iDS})$ has been arbitrarily chosen.

The method consists of superposing the graphs of $\dot{\alpha}_{iDS,max}(\dot{\alpha}_{fDS})$ $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{fDS})$ and $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$. The two first functions will be calculated by a numerical integration at each point whereas the third results from the evaluation of one integral only (see eqs. (48) and (49) in Appendix C). We present an example of these graphs in Figure 8. The case of this figure is in fact that obtained by the optimization process presented in Section 3.5.

We give here some basic comments about Figure 8, but the reader could refer to Miossec (2004) for more comprehensive comments. This graph first allows us to see all the possible cyclic motions for a given walking motion, when considering different dynamics for α. Cyclic motions are represented in this graph by points on the equation $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$. From Propositions 3 and 4, a cyclic stable motion must satisfy $\dot{\alpha}_{c,iDS} \in]\dot{\alpha}_{iDS,min}, \dot{\alpha}_{iDS,max}[$. Therefore, cyclic stable motions will be represented by points on the graph $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$ when $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{fDS}) < a(\dot{\alpha}_{fDS}) < \dot{\alpha}_{iDS,max}(\dot{\alpha}_{fDS})$; that is, when the graph of function $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$ is between the graphs of functions $\dot{\alpha}_{iDS,max}(\dot{\alpha}_{fDS})$ and $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{fDS})$. We can then easily see the zone of existence of cyclic motions. In fact, in the case presented, it corresponds to the admissible zone of single support, denoted D_{gfDS}.

It is also possible to read the convergence domain in one step and the global convergence domain for both Poincaré representations of $\dot{\alpha}_{iDS}$ and $\dot{\alpha}_{fDS}$. It also allows us to see that here the convergence domains are limited by constraints in single support. It would also be possible to determine the number of steps needed to converge from the extremities of the attraction domains. In particular, since the curve $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$ is very close to $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{fDS})$ the convergence to cyclic motion will be slow when the speed of the walk is too quick compared to the speed convergence when the speed of the biped is too slow. This leads to the idea it would be good to equilibrate distance between curve $\dot{\alpha}_{iDS} = a(\dot{\alpha}_{fDS})$ and curves $\dot{\alpha}_{iDS,max}(\dot{\alpha}_{fDS})$ and $\dot{\alpha}_{iDS,min}(\dot{\alpha}_{fDS})$ and enlarge the possible zone as far as possible.

All the criteria presented here could be used in the optimization process presented in Section 3.5 in order to obtain more stable motions and quicker convergence.

We have given conditions for the existence of a control law that can give a stable walk with a one-step convergence zone. We have shown that the control law (12) with a reference motion that satisfies constraints allows us to obtain stability when possible and to obtain the largest one-step convergence zone possible. We have given a graphical representation to choose the speed of evolution of α during the walk from a stability point of view. However, we have not addressed the choice of motion $\alpha_{DS}(t)$ among an infinity. In fact, this choice has no influence on the stability of the motion with the proposed control law. All that is important is that it must satisfy constraints (13). The choice of this motion of α in double support can then be done only in order to minimize an energy criterion, for example. We have obtained such a reference motion by the optimization process presented in Section 3.5.

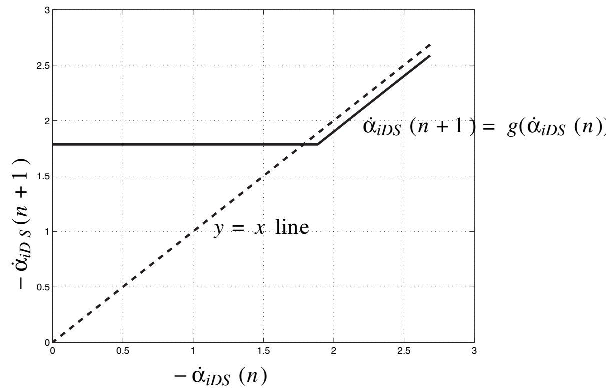

Fig. 9. Representation of the Poincaré map at the beginning of double support.

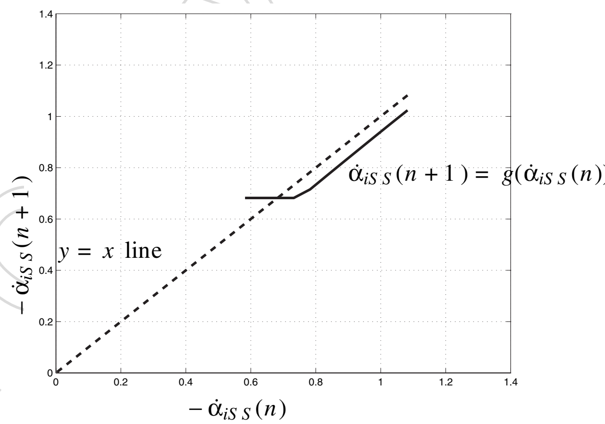

Fig. 10. Representation of the Poincaré map at the beginning of single support.

6. Simulation Results

We present here some stability results for a reference motion we obtained by the optimization process presented in Section 3.5. This reference motion satisfies the constraints in double support.

We show here the Poincaré return map of α_{iSS} (Figure 10) and of α_{iDS} (Figure 9) with the control law (12) presented. For better representation of the Poincaré return map, we represent the opposite of $\dot{\alpha}$, since $\dot{\alpha} < 0$. These two Poincaré maps do not contain more information than Figure 8, but the information is clearer.

We can see that in both Poincaré maps the null slope zone is large. This zone guarantees the convergence from the attraction domain in a finite number of steps. The attraction domain is limited on the Poincaré return map of $\dot{\alpha}_{iSS}$ for small velocities by the fall back of the biped during the single-support phase (because the initial speed is too small) and for high velocities by the torque limitation in the single-support phase. The attraction domain for the Poincaré map of $\dot{\alpha}_{iDS}$ is not limited in small velocities. This means that the biped can start walking from stop. Furthermore, due to the large one-step convergence zone, this start phase leads directly to the cyclic motion in one step. The limitation at high speed is the same as in the other Poincaré return map. It is also interesting to see that the zone of convergence in the one step of the $\dot{\alpha}_{iDS}$ Poincaré return map is far larger for small speeds than high speeds. We can see in Figure 8 that it is because the distance between the cyclic motion and the constraints is very unequal.

The motion studied here has been optimized only with an energy criterion, without stability constraints. The work presented has only been to constat the good stability of this motion. In future work, it would be interesting to take into account the stability in the motion optimization.

7. Conclusion

We have studied the dynamics of α, which are the less stable dynamics, since they are not controlled during single support.

We have given an efficient control law in double support for the stabilization of these dynamics. We have shown that an infinity of cyclic motions exists. It is important to note that those cyclic motions must satisfy some constraints. We determined the one-step convergence zone to a cyclic motion and showed that this zone can always exist when the cyclic motion is stable. We have seen in simulation results for a given cyclic motion that the one-step convergence zone is quite large and even allows start from stop, and that the global attraction domain is even larger. The existence of the one-step convergence zone is due to the presence of the overactuated double-support phase, well exploited with the control law presented. We thus see the interest of the double-support phase in the walking gait of a biped to obtain a good stability. However, a drawback is that the control used is sensitive to model errors, since the acceleration constraints depend on the model. To increase robustness, we could increase the security distance with constraints, or use a force sensor measure. Furthermore, the possible commutation of walking speed by changing the dynamics in α could be used to stop the biped, or finally could be exploited in order to commute easier between different motions obtained by optimization. Another perspective is to include the size of the one-step convergence zone or the global attraction domain as the optimization criteria in the motion generation, in order to increase stability.

Appendix A: Calculation of Vector *b*

During the walking gait the legs have a symmetrical role from one half-step to the next half-step. Then, from a practical point of view, this walking gait can be studied only on one half-step if the exchange of the leg role is taken into account. A half-step is then composed of the sequence of the single-support phase, the impact, the exchange of the leg role, and the double-support phase. After the impact and before the exchange of the leg role, let the velocity vectors be $\dot{X}^+_{int}$. After the exchange of the leg role, let the velocity vector be $\dot{X}^+$. The exchange of the leg role leads to the following relations for each coordinate of vectors $\dot{X}^+$ and $\dot{X}^+_{int}$:

$$\begin{cases} \dot{\alpha}^+ = \dot{\alpha}^+_{int} + \dot{\delta}^+_{1int} + \dot{\delta}^+_{3int} - \underset{.}{o}t\delta^+_{4int} \\ \dot{\delta}^+_1 = \dot{\delta}^+_{4int} \\ \dot{\delta}^+_2 = \dot{\delta}^+_{2int} + \dot{\delta}^+_{3int} \\ \dot{\delta}^+_3 = -\dot{\delta}^+_{3int} \\ \dot{\delta}^+_4 = \dot{\delta}^+_{1int} \\ \dot{x}^+_t = \dot{x}^+_{tint} \\ \dot{y}^+_t = \dot{y}^+_{tint}. \end{cases} \tag{21}$$

Let us introduce matrix J to define the exchange of the leg role. We have then the following relation between $\dot{X}^+$ and $\dot{X}^+_{int}$

$$\dot{X}^+ = J\dot{X}^+_{int}, \tag{22}$$

where matrix $J(7\times 7)$, composed of 0 and 1, is deduced from eq. (21):

$$J = \begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

We have the following relations to describe the impulsive impact:

$$A(\dot{X}^+_{int} - \dot{X}^-) = D_1 I_{R_1} + D_2 I_{R_2} \tag{24}$$

$$\begin{cases} D_1^{\mathrm{T}}\dot{X}^+_{int} = 0 \\ D_2^{\mathrm{T}}\dot{X}^+_{int} = 0 \end{cases} \tag{25}$$

$$D_1^{\mathrm{T}}\dot{X}^- = 0. \tag{26}$$

System (25) means we suppose a double support after impact. The last vector–matrix equation (26) means the point foot of the stance leg is fixed just before impact with $D_1(7\times 2)$. We can rewrite eq. (26),

$$D_{1_{15}}^{\mathrm{T}}\dot{q}^- + D_{1_{67}}^{\mathrm{T}}\begin{bmatrix} \dot{x}_t^- \\ \dot{y}_t^- \end{bmatrix} = 0 \tag{27}$$

where $\begin{bmatrix} \dot{x}_t^- & \dot{y}_t^- \end{bmatrix}^{\mathrm{T}}$ represents the velocity of the trunk mass center just before the impact. $D_{1_{67}}^{\mathrm{T}}$ is composed of all lines and columns 6 and 7 of matrix D_1^{T}, and $D_{1_{15}}^{\mathrm{T}}$ is composed of all lines and columns 1–5 of matrix D_1^{T}. $D_{1_{67}}^{\mathrm{T}}$ is in fact equal to a (2×2) identity matrix. Equation (27) can then be rewritten simply as

$$\begin{bmatrix} \dot{x}_t^- \\ \dot{y}_t^- \end{bmatrix} = -D_{1_{15}}^{\mathrm{T}}\dot{q}^-. \tag{28}$$

If we assume a perfect tracking of the joint angles $\delta_i{=}\delta_i(\alpha)$ we have the four relations, $\dot{\delta}_i(\alpha,\dot{\alpha}) = \partial\delta_i/\partial\alpha\dot{\alpha}$ $(i = 1, ..., 4)$. Then with these four relations and using the vector–matrix equation (27) we can write the velocity vector $\dot{X}^-$

$$\dot{X}^- = \begin{bmatrix} I_{5\times 5} \\ -D_{1_{15}}^{\mathrm{T}} \end{bmatrix}\dot{q}^- \tag{29}$$

with

$$\dot{q}^- = \begin{bmatrix} 1 \\ \dfrac{\partial\delta_1}{\partial\alpha} \\ \dfrac{\partial\delta_2}{\partial\alpha} \\ \dfrac{\partial\delta_3}{\partial\alpha} \\ \dfrac{\partial\delta_4}{\partial\alpha} \end{bmatrix} \dot{\alpha}^- = q^*\dot{\alpha}^- \tag{30}$$

where $I_{5\times5}$ is a (5×5) identity matrix.

Then $\dot{X}^-$ is proportional to $\dot{\alpha}^-$.

From impact equations (24), (25), and (26) and eqs. (23), (29), and (30) it is possible to obtain vectors $\dot{X}^+$, I_{R_1} and I_{R_2} as a function of $\dot{\alpha}^-$. First we solve the impact equation to determine $\dot{X}^+_{int}$, I_{R_1}, and I_{R_2} as functions of $\dot{X}^-$. From eqs. (24), (25), and (26) we can obtain

$$A_{impact} \begin{bmatrix} \dot{X}^+_{int} \\ I_{R_1} \\ I_{R_2} \end{bmatrix} = \begin{bmatrix} A\dot{X}^- \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

where

$$A_{impact} = \begin{bmatrix} A & -D_R \\ D_R^{\mathrm{T}} & 0 \end{bmatrix} \qquad \text{and}$$

$$D_R^{\mathrm{T}}(4\times7) = \begin{bmatrix} D_1 & D_2 \end{bmatrix}^{\mathrm{T}}. \tag{32}$$

Then[3]

$$\begin{bmatrix} \dot{X}^+_{int} \\ I_{R_1} \\ I_{R_2} \end{bmatrix} = A^{-1}_{impact} \begin{bmatrix} A\dot{X}^- \\ 0 \\ 0 \end{bmatrix}. \tag{33}$$

We can write the relation between $\dot{X}^+$ and $\dot{\alpha}^-$ from eq. (33) using eqs. (23), (29), and (30),

$$\dot{X}^+ = J(A^{-1}_{impact})_{17\,17} A \begin{bmatrix} I_{5\times5} \\ -D^{\mathrm{T}}_{1_{15}} \end{bmatrix} q^*\dot{\alpha}^- \tag{34}$$

where $(A^{-1}_{impact})_{17\,17}$ is the submatrix of A^{-1}_{impact} corresponding to lines 1–7 and columns 1–7.

Finally, we can conclude that the expression for the coefficient b is the first component of $\dot{X}^+$

$$\dot{\alpha}^+ = b\dot{\alpha}^- \tag{35}$$

with

$$b = J_1(A^{-1}_{impact})_{17\,17} A \begin{bmatrix} I_{5\times5} \\ -D^{\mathrm{T}}_{1_{15}} \end{bmatrix} q^* \tag{36}$$

where J_1 is the first line of J.

3. A mathematical study of the invertibility of matrix A_{impact} has proved that it is singular only when D_R is not of rank 4. This occurs when $\forall \quad i = 1, \ldots, 4 \quad \delta_i = 0$ or π, which means that all the links of the legs are in the same direction. The physical interpretation of this situation is that there is an infinity of possible impulsive ground reactions at impact corresponding to solutions with different repartitions of impulsive ground reactions between both feet along the direction of the legs. This singularity is never obtained since it corresponds to an impossible situation for a walking motion.

Appendix B: Calculation of Matrices A_α and H_α

We assume that the biped is in double support and that the tracking of the joints is perfect, $\delta_i = \delta_i(\alpha)$ $(i = 1, 2)$. First, using geometric relations these assumptions allow us to write vector $X = \begin{bmatrix} \alpha & \delta_1 & \delta_2 & \delta_3 & \delta_4 & x_t & y_t \end{bmatrix}^{\mathrm{T}}$ explicitly as a function of α:

$$X = X(\alpha). \tag{37}$$

Let be matrix $D_R(7\times4) = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$. In double support, the velocity of both leg tips being zero we can obtain $\dot{X}$ as a function of α, and $\dot{\alpha}$ from the second relation of eq. (4)

$$D_R^{\mathrm{T}}(q)\dot{X} = 0. \tag{38}$$

Let us introduce the new matrices $D_{R_{13}}(3\times4)$ corresponding to lines 1–3 of D_R and $D_{R_{47}}(4\times4)$ corresponding to lines 4–7 of D_R to rewrite eq. (38)

$$D^{\mathrm{T}}_{R_{13}} \begin{bmatrix} \dot{\alpha} \\ \dot{\delta}_1 \\ \dot{\delta}_2 \end{bmatrix} + D^{\mathrm{T}}_{R_{47}} \begin{bmatrix} \dot{\delta}_3 \\ \dot{\delta}_4 \\ \dot{x} \\ \dot{y} \end{bmatrix} = 0 \tag{39}$$

where $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ represents the velocity of the trunk mass center. Using this vector–matrix equation (39), the inverse matrix[4] of $D^{\mathrm{T}}_{R_{47}}$ and the relation $\dot{\delta}_i(\alpha, \dot{\alpha}) = \partial\delta_i/\partial\alpha\dot{\alpha}$ $(i = 1, 2)$, we obtain

$$\dot{X}(\alpha, \dot{\alpha}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \multicolumn{3}{c}{-D^{T-1}_{R_{47}} D^{\mathrm{T}}_{R_{13}}} \end{bmatrix} \begin{bmatrix} 1 \\ \dfrac{\partial\delta_1}{\partial\alpha} \\ \dfrac{\partial\delta_2}{\partial\alpha} \end{bmatrix} \dot{\alpha}. \tag{40}$$

Let matrix $H_c(7\times1) = \begin{bmatrix} H_{c1}(q, \dot{q}) \\ H_{c2}(q, \dot{q}) \end{bmatrix}$. The third relation of eq. (4) for $j = 1$ and $j = 2$ can be merged in only one vector–matrix equation,

$$D_R^{\mathrm{T}}(q)\ddot{X} + H_c(q, \dot{q}) = D^{\mathrm{T}}_{R_{13}} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{bmatrix} + D^{\mathrm{T}}_{R_{47}} \begin{bmatrix} \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{x} \\ \ddot{y} \end{bmatrix} + H_c(q, \dot{q}) = 0. \tag{41}$$

4. A mathematical study of the invertibility of matrix $D_{R_{47}}$ has proved that it is singular only when $\delta_4 = 0$ or π, which corresponds to a completely stretched or folded up leg. Those singular cases are classical for manipulator robot. Those singularities are avoided by the motion generation process; then the reference motions obtained are far from it. In the present case where the joint reference trajectories $\delta_i(\alpha)$ $i = 1, \ldots, 4$ are exactly tracked, those singularities are then avoided.

Using eq. (41), $X(\alpha)$, $\dot{X}(\alpha,\dot{\alpha})$, the inverse matrix of $D^{\mathrm{T}}_{R_{47}}$ and the relation $\ddot{\delta}_i(\alpha,\dot{\alpha},\ddot{\alpha}) = \partial\delta_i/\partial\alpha\ddot{\alpha} + \partial^2\delta_i/\partial\alpha^2\dot{\alpha}^2$, we can write $\ddot{X}(\alpha,\dot{\alpha},\ddot{\alpha})$,

$$\ddot{X}(\alpha,\dot{\alpha},\ddot{\alpha}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \multicolumn{3}{c}{-D^{T-1}_{R_{47}} D^{\mathrm{T}}_{R_{13}}} \end{bmatrix} \left[\begin{bmatrix} 1 \\ \dfrac{\partial\delta_1}{\partial\alpha} \\ \dfrac{\partial\delta_2}{\partial\alpha} \end{bmatrix} \ddot{\alpha} + \begin{bmatrix} 0 \\ \dfrac{\partial^2\delta_1}{\partial\alpha^2} \\ \dfrac{\partial^2\delta_2}{\partial\alpha^2} \end{bmatrix} \dot{\alpha}^2 \right] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -D^{T-1}_{R_{47}} H_c(\alpha,\dot{\alpha}) \end{bmatrix}. \tag{42}$$

Then, using the previous result (42) in the dynamic model (1) of the biped, we obtain

$$A_\alpha\ddot{\alpha} + H_\alpha(\alpha,\dot{\alpha}) = D_\Gamma\Gamma + D_R \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \tag{43}$$

where $A_\alpha = A(\alpha) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \multicolumn{3}{c}{-D^{T-1}_{R_{47}} D^{\mathrm{T}}_{R_{13}}} \end{bmatrix} \begin{bmatrix} 1 \\ \dfrac{\partial\delta_1}{\partial\alpha} \\ \dfrac{\partial\delta_2}{\partial\alpha} \end{bmatrix}$

and $H_\alpha = H(\alpha,\dot{\alpha}) + A(\alpha) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \multicolumn{3}{c}{-D^{T-1}_{R_{47}} D^{\mathrm{T}}_{R_{13}}} \end{bmatrix} \begin{bmatrix} 0 \\ \dfrac{\partial^2\delta_1}{\partial\alpha^2} \\ \dfrac{\partial^2\delta_2}{\partial\alpha^2} \end{bmatrix} \dot{\alpha}^2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -D^{T-1}_{R_{47}} H_c(\alpha,\dot{\alpha}) \end{bmatrix}.$

Appendix C: Expressions of Matrices B, C and D

These three matrices are calculated by inverting[5] eq. (14)

$$\begin{aligned} B(\alpha) &= D_{tot}(\alpha)^{-1} A_\alpha(\alpha) \\ C(\alpha,\dot{\alpha}) &= D_{tot}(\alpha)^{-1} H_\alpha(\alpha,\dot{\alpha}) \\ D(\alpha) &= D_{tot}(\alpha)^{-1} D_{2x}(\alpha) \end{aligned} \tag{44}$$

with $D_{tot} = \begin{bmatrix} D_1 & D_{2z} & D_\Gamma \end{bmatrix}$, where D_{2x} is the first column of D_2 and D_{2z} is the second column of D_2.

5. A mathematical study of the invertibility of matrix D_{tot} has proved that it is singular only when both feet have the same position along the x-axis. A physical interpretation of this situation is that the repartition between both feet of the vertical ground reactions can be chosen as we do yet for the horizontal components. This singular case will never occur in our study since it is incompatible with a walking motion.

Appendix D: Calculation of Matrices E, F, G and P

Inequation (13) can be rewritten in term of matrices as

$$M \begin{bmatrix} R_{1x} \\ R_{1y} \\ R_{2y} \\ \Gamma \end{bmatrix} + N R_{2x} + P \leq 0 \tag{45}$$

with $M = \begin{bmatrix} 0 & -1 & 0 & \\ 0 & 0 & -1 & \\ -1 & -f_{max} & 0 & \\ 0 & 0 & -f_{max} & 0_{6\times4} \\ 1 & -f_{max} & 0 & \\ 0 & 0 & -f_{max} & \\ \multicolumn{3}{c}{0_{4\times3}} & -I_{4\times4} \\ \multicolumn{3}{c}{0_{4\times3}} & I_{4\times4} \end{bmatrix}$ $N = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0_{8\times1} \end{bmatrix}$ $P = \begin{bmatrix} R_{1y,min} \\ R_{2y,min} \\ 0_{6\times1} \\ -\Gamma_{max} 1_{8\times1} \end{bmatrix}$. Here, $0_{i\times j}$ is an $(i \times j)$ matrix composed of zeros, $1_{i\times j}$ is an $(i \times j)$ matrix composed of ones, and $I_{i\times i}$ is an $(i \times i)$ identity matrix.

By replacing R_{1x}, R_{1y}, R_{2y}, and Γ in inequation (45) with their expression obtained from eq. (15) we obtain inequation (16) with,

$$\begin{aligned} E &= MB(\alpha) \\ F &= MC(\alpha,\dot{\alpha}) \\ G &= MD(\alpha) + N \end{aligned} . \tag{46}$$

Appendix E: Calculation of Function $a()$

The results presented here are obtained from Chevallereau, Formal'sky, and Djoudi (2003) and are adapted to our motion definition. In order to determine the relation between $\dot{\alpha}_{iDS}$ and $\dot{\alpha}_{fDS}$, we first start from the relation for single support between $\dot{\alpha}_{iSS}$ and $\dot{\alpha}_{fSS}$

$$\dot{\alpha}_{fSS} = -\frac{\sqrt{\psi(\alpha_{fSS}) + f(\alpha_{iSS})^2\dot{\alpha}^2_{iSS}}}{f(\alpha_{fSS})} \tag{47}$$

$$\psi(\alpha) = -2\int_{\alpha_{iss}}^{\alpha} Mg(x_G(\mu) - x_S)\mathrm{d}\mu. \tag{48}$$

Then the impact equation (9) gives the relation between $\dot{\alpha}_{iDS}$ and $\dot{\alpha}_{fSS}$, and we have $\dot{\alpha}_{iSS} = \dot{\alpha}_{fDS}$ since there is no impact nor leg exchange between the end of double support and the beginning of single support. Finally, we obtain

$$\dot{\alpha}_{iDS} = -b\frac{\sqrt{\psi(\alpha_{fSS}) + f(\alpha_{iSS})^2\dot{\alpha}_{fDS}^2}}{f(\alpha_{fSS})}. \tag{49}$$

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References

Aoustin, Y., and Formal’sky, A. M. 2003. Control design for a biped: reference trajectory based on driven angles as functions of the undriven angle. *Journal of Computer and System Sciences International* 42(4):159–176.

Cheng, M. M. and Lin, C. S. 1996. Measurement of robustness for biped locomotion using a linearized Poincaré Map. *Robotica* 42(4):253–259.

Chevallereau, C., Formal’sky, A., and Djoudi, D. 2003. Tracking of a joint path for the walking of an underactuated biped. *Robotica* 22(1):15–28.

Chevallereau, C., Abba, G., Aoustin, Y., Plestan, F., Westervelt, A. R., Canudas de Wit, C., and Grizzle, J. W. 2003. RABBIT: a testbed for advanced control theory. *IEEE Control Systems Magazine* 23(5):57–78.

Dantzig, G. 1963. *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ.

Devy, M., Espiau, B., Genot, F., Ghallab, M., Lamiraux, F., Laumond, J. P., Morin, P., Rives, P., Samson, C., and Sekhavat, S. 2001. La robotique mobile. *Hermes IC2*, Paris, France, pp. 345.

Formal’sky, A. M. 1982. *Locomotion of Anthropomorphic Mechanisms*, Nauka, Moscow (in Russian).

Formal’sky, A. M. 1997. Ballistic locomotion of a biped. Design and control of two biped machines. *Human and Machine Locomotion*, CISM, Udine, Italy, A. Morecki and K. Waldron, editors, Springer-Verlag, Berlin.

Goswami, A. 1999 Foot rotation indicator point: a new gait planning tool to evaluate postural stability of biped robots. *Proceedings of the IEEE Conference on Robotics and Automation*, Detroit, MI, pp. 47–52.

Goswami, A., Espiau, B., and Keramane, A. 1997. Limit cycles in a passive compass gait biped and passivity-mimicking control laws. *Journal of Autonomous Robots* 4(3):273–286.

Grishin, A. A, Formal’sky, A. M., Lensky, A. V., and Zhitomirsky, S. V. 1994. Dynamic walking of a vehicle with two telescopic legs controlled by two drives. *International Journal of Robotics Research* 13(2):137–147.

Grizzle, J. W., Abba, G. and Plestan, F. 2001. Asymptotically stable walking for biped robots: analysis via systems with impulse effects. *IEEE Transactions on Automatic Control* 46(1):51–64.

Hirai, K., Hirose, M., Haikawa, Y., and Takenaka, T. 1998. The development of Honda Humanoid Robot. *Proceedings of the IEEE Conference on Robotics and Automation*, Leuven, Belgium, pp. 1321–1326.

Löffler, K., Gienger, M., and Pfeiffer, F. 2003. Sensors and control concept of walking “Johnnie”. *International Journal of Robotics Research* 22(3):229–239.

Miossec, S. 2004. Contribution à l’étude de la marche d’un bipède. Ph.D. Thesis, Ecole Centrale de Nantes et Université de Nantes, France (in French).

Miossec, S. and Aoustin, Y. 2002a. Mouvement de marche composé de simple et double supports pour un robot bipède planaire sans pieds. *Proceedings of CIFA 2002*, France (in French).

Miossec, S., and Aoustin, Y. 2002b. Walking gait composed of single and double supports for a planar biped without feet. *Proceedings of the 5th International Conference on Climbing and Walking Robots (CLAWAR)*, Paris, France, September 25–27, pp. 767–774.

Roussel, L., Canudas-de-Wit, C., and Goswami, A. 1998. Generation of energy optimal complete gait cycles for biped. *Proceedings of the IEEE Conference on Robotics and Automation*, Leuven, Belgium, pp. 2036.

Vukobratovic, M., Borovac, B., Surla, D., and Stokic, D. 1990. *Biped Locomotion*, Springer-Verlag, Berlin.

Zonfrilli, F., Oriolo, G., and Nardi, D. 2002. A biped locomotion strategy for the quadruped robot Sony ERS-210. *Proceedings of the IEEE Conference on Robotics and Automation*, Washington, DC, May 11–15, pp. 2768–2774.